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Quantum Systems on Linear Groups

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Discussed are quantized dynamical systems on orthogonal and affine groups. The special stress is laid on geodetic systems with affinely-invariant kinetic energy operators. The resulting formulas show that such models may be useful in nuclear and hadronic dynamics. They differ from traditional Bohr–Mottelson models where $SL(n, \mathbb{R})$ is used as a so-called non-invariance group. There is an interesting relationship between classical and quantized integrable lattices.

KEY WORDS: quantized affine systems; quantized rigid body; multi-valued wave functions.

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1. INTRODUCTION: MULTI-VALUED WAVE FUNCTIONS

In this section we deal with the simple Schrödinger quantization, i.e., with wave mechanics on differential manifolds.

Let Q be a configuration space, i.e., differential manifold of dimension f (the number of classical degrees of freedom). If it is endowed with some positive volume measure μ, then the wave functions may be considered as complex scalar fields $\Psi : Q \to \mathbb{C}$. The corresponding scalar product is given by

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \overline{\Psi_1(q)} \Psi_2(q) \; d\mu(q), \tag{1}$$

and our Hilbert space is meant as $L^2(Q, \mu)$. Usually, μ comes from some Riemannian structure (Q, g) and then $d\mu(q) = \sqrt{|\det[g_{ij}]|} dq^1 \cdots dq^f$. As shown and discussed by Mackey (1963) one can do quite well without any μ if wave amplitudes are considered not as scalars but instead as complex 1/2-weight densities, and then simply

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi_1(q)} \Psi_2(q) \; dq^1 \cdots dq^f. \tag{2}$$

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The well-known text-book conditions on wave functions (does not matter scalars or densities) are as follows: (i) Ψ is to be one-valued all over Q, (ii) Ψ is continuous with derivatives even at potential jumps. This is justified by probabilistic interpretation of $\bar{\Psi}\Psi$, probability conservation, Sturm–Liouville theory, essential self-adjointness of certain operators.

There were, however, some arguments by Pauli and Reiss that the one-valuedness is not the basic postulate of quantum mechanics. There are some path-dependence phenomena and problems with globalization of local solutions in multiply-connected Q's.

What is a reasonable "multi-valuedness" in this context? The one that takes $\bar{Q}$, the universal covering manifold of Q with the projection $\pi : \bar{Q} \to Q$, and admits wave functions defined rather on $\bar{Q}$ than on Q. This has also to do with projective representations used in quantum mechanics. The procedure seems to be reasonable when the co-images $\pi^{-1}(q)$ are finite sets.

One of possible examples is the system of identical particles, when removing the diagonals from the Cartesian product and performing appropriate identifications (taking quotients) one damages drastically topological structure of the configuration space.

2. RIGID BODY AND DOUBLY-VALUED WAVE FUNCTIONS

Another example close to our subject is the rigid body, where $Q \simeq \mathrm{SO}(3, \mathbb{R})$ and $\bar{Q} \simeq \mathrm{SU}(2)$. Then the projection π is $2:1$, i.e., for any $u \in \mathrm{SU}(2)$, $\pm u$ are projected onto the same element of $\mathrm{SO}(3, \mathbb{R})$. So, there is a natural hope that the system of spin-less particles bounded by an appropriate potential making it (approximately) rigid may show half-integer rotational angular momentum (Arsenović *et al.*, 1995a,b; Barut *et al.*, 1992; Barut and Rączka, 1977; Pauli, 1939; Reiss, 1939). By "spin" in the earlier "spin-less" we mean the usual intrinsic angular momentum treated as something primary, non-explained in the usual rotational sense. By the way, non-explained need not mean non-explainable; some idea about fundamental particles as rigid or deformable quantized tops is often coming back to physics in spite of its exotic character.

Let $D^j : \mathrm{SU}(2) \to \mathrm{GL}(2j+1, \mathbb{C})$ denote irreducible unitary representations of SU(2); j runs over non-negative integers and half-integers starting from zero (Wigner matrices). Obviously, $D^j(u)^+ = D^j(u)^{-1} = D^j(u^{-1})$ and $D^j(-u) = (-1)^{2j} D^j(u)$. For integer js D^j is projectable to $\mathrm{SO}(3, \mathbb{R})$; for non-integer ones one deals with "two-valued" representations of $\mathrm{SO}(3, \mathbb{R})$. Traditional symbols for matrix elements are $D^j_{m,m'}(u)$, where $m, m' = -j, -j+1, \ldots, j-1, j$. According to the Peter–Weyl theorem the wave amplitudes on SU(2) may be expanded into series:

$$\Psi(u) = \sum_{j=0}^{\infty} \mathrm{Tr}\left(c^j D^j(u)\right), \qquad c^j \in \mathrm{L}(2j+1, \mathbb{C}). \tag{3}$$

Statistical interpretation in SO(3, $\mathbb{R}$), $\bar{\Psi}\Psi(-u) = \bar{\Psi}\Psi(u)$, imposes the "superselection" rule: D^j with different "halfness" of j cannot be superposed. There are two disjoint situations: the "fermionic" and "bosonic" ones,

$$\Psi_f(u) = \sum_{i=1}^{\infty} \mathrm{Tr}\left(c^{i-1/2} D^{i-1/2}(u)\right), \qquad \Psi_b(u) = \sum_{j=0}^{\infty} \mathrm{Tr}(c^j D^j(u)). \tag{4}$$

Left and right regular translations are described respectively as

$$\Psi'(u) := \Psi(vu), \quad c'^j := c^j D^j(v), \qquad \Psi'(u) := \Psi'(uv), \quad c'^j := D^j(v)c^j. \tag{5}$$

Let $\mathbf{L}_a$, $\mathbf{R}_a$ be differential operators generating respectively left and right regular translations,

$$\Psi(u(\bar{\epsilon})u(\bar{k})) = \Psi(u(\bar{k})) + \epsilon^a \mathbf{L}_a \Psi(\bar{k}) + o(\epsilon), \tag{6}$$

$$\Psi(u(\bar{k})u(\bar{\epsilon})) = \Psi(u(\bar{k})) + \epsilon^a \mathbf{R}_a \Psi(\bar{k}) + o(\epsilon), \tag{7}$$

where the rotation vector $\bar{k}$ is used, i.e., canonical coordinates of the first kind, $u(\bar{k}) = \exp\left(-(i/2)k^a \sigma_a\right)$, $|k| \leq 2\pi$, and σ_a are, obviously, Pauli matrices. The laboratory and co-moving representations of spin operators are given respectively by the following expressions: $\mathbf{S}_a = (\hbar/i)\mathbf{L}_a$, $\hat{\mathbf{S}}_a = (\hbar/i)\mathbf{R}_a$. Their quantum Poisson brackets are as follows:

$$\frac{1}{i\hbar}[\mathbf{S}_a, \mathbf{S}_b] = \varepsilon_{ab}{}^c \mathbf{S}_c, \qquad \frac{1}{i\hbar}[\hat{\mathbf{S}}_a, \hat{\mathbf{S}}_b] = -\varepsilon_{ab}{}^c \hat{\mathbf{S}}_c, \qquad \frac{1}{i\hbar}[\mathbf{S}_a, \hat{\mathbf{S}}_b] = 0. \tag{8}$$

Obviously,

$$D^j(u(\bar{k})) = \exp\left(\frac{i}{2} k^a S^j_a\right), \tag{9}$$

where S^j are Wigner matrices for the jth angular momentum (Rose, 1995),

$$\left(S^j_1\right)^2 + \left(S^j_2\right)^2 + \left(S^j_3\right)^2 = \hbar^2 j(j+1)\mathrm{Id}_{2j+1} \tag{10}$$

(Casimir invariant properties). The action of spin operators on D^j is algebraized: $\mathbf{S}_a D^j = S^j_a D^j$, $c^j \mapsto c^j S^j_a$ and $\hat{\mathbf{S}}_a D^j = D^j S^j_a$, $c^j \mapsto S^j_a c^j$. In particular,

$$\mathbf{S}_3 D^j_{m,m'} = \hbar m D^j_{m,m'}, \qquad \hat{\mathbf{S}}_3 D^j_{m,m'} = \hbar m' D^j_{m,m'},$$

$$((\mathbf{S}_1)^2 + (\mathbf{S}_2)^2 + (\mathbf{S}_3)^2)D^j = ((\hat{\mathbf{S}}_1)^2 + (\hat{\mathbf{S}}_2)^2 + (\hat{\mathbf{S}}_3)^2)D^j = \hbar^2 j(j+1)D^j.$$

Hamiltonian has the following form:

$$\mathbf{H} = \mathbf{T} + \mathbf{V} = \frac{1}{2I_1}(\hat{\mathbf{S}}_1)^2 + \frac{1}{2I_2}(\hat{\mathbf{S}}_2)^2 + \frac{1}{2I_3}(\hat{\mathbf{S}}_3)^2 + V(u). \tag{11}$$

The above **T** is invariant under left regular translations (spatial rotations). For the spherical top, $I_1 = I_2 = I_3$, it is also invariant under right regular translations (material rotations). For the symmetric top, $I_1 = I_2$, it is invariant under SO(2, $\mathbb{R}$)-right translations (material rotations about the third main axis of inertia). For the non-degenerate case, $I_1 \neq I_2 \neq I_3$, there are no material symmetries. Nevertheless, for any ratio of inertial moments the metric tensor underlying the kinetic energy form is left invariant and so is its induced Riemannian volume. Therefore, this volume is simply proportional to the Haar measure on SU(2) (SO(3, $\mathbb{R}$)), so it may be directly obtained without embarrassing manipulations on the anisotropic metric tensor.

It is seen that for the quantized geodetic case, $V = 0$, the problem is fully algebraized and the differential eigenequation $\mathbf{T}\Psi = E\Psi$ splits into the family of algebraic ones for c^j-matrices:

$$\left(\frac{1}{2I_1}\left(S_1^j\right)^2 + \frac{1}{2I_2}\left(S_2^j\right)^2 + \frac{1}{2I_3}\left(S_3^j\right)^2\right) c^j = E^j c^j. \tag{12}$$

For the symmetric top, $I_1 = I_2 = I$, $I_3 = K$, and even more so for the spherical one, $I = K$, $D^j_{m,m'}$ are eigenfunctions of the basic operators and of **T** itself, and the eigenvalues may be immediately found (the degeneracy structure is explicitly seen):

$$E^j_{m'} = \frac{\hbar^2 j(j+1)}{2I} + \left(\frac{1}{2K} - \frac{1}{2I}\right)\hbar^2 m'^2. \tag{13}$$

If V exists and is a simple combination of D^j-functions, the problem may be also reduced to algebraic equations on the basis of Clebsch–Gordan series; however, as a rule the resulting algebraic system is infinite (thus, in general, effective only when some approximate truncation is possible).

In the papers (Sławianowski, 1980; Sławianowski and Słomiński, 1980) we considered Bertrand-type models for the spherical top, i.e., isotropic models with all trajectories closed. One of them was degenerate oscillator, $V = 2\kappa \tan^2(\varphi/2)$, $\kappa > 0$, where φ denotes the angle of deflection from the equilibrium orientation. Due to the singularity at $\varphi = \pi$ (infinite potential barrier) this is, as a matter of fact, the problem on SO(3, $\mathbb{R}$), the usual rigid body configuration space. However, in the limit $\kappa \to 0$, we obtain the free rigid body with the half-integer angular momentum admitted. Just another illustration of the idea of "half-integerness" for extended systems.

Everything said above remains valid for the abstract n-dimensional rigid body, $n > 2$, where the 2 : 1 covering of SO(n, $\mathbb{R}$) is the group Spin(n).

3. QUANTIZED AFFINE BODIES AND DOUBLY-VALUED WAVE FUNCTIONS

Let us now discuss quantization of an affinely-rigid body (Sławianowski, 1982, 1988) without translations. By the affinely-rigid body we mean such one that all affine relationships between its constituents are preserved during its motion (rigid in the sense of affine geometry). So, we deal with Schrödinger wave mechanics on $\mathrm{GL}^+(n, \mathbb{R})$ or $\mathrm{SL}(n, \mathbb{R})$ in the incompressible case. For $n = 3$ such degrees of freedom are used in the droplet model of atomic nuclei (Bohr and Mottelson, 1975). However, only kinematics is there directly ruled by $\mathrm{SL}(3, \mathbb{R})$, the dynamics is not invariant under this group. Rather, $\mathrm{SL}(3, \mathbb{R})$ is there the dynamical non-invariance group which enables one to investigate the energy spectrum in terms of some ladder procedure. The whole beauty and analytical usefulness of group-theoretic degrees of freedom are then lost. We are going to construct kinetic energies (metric tensors) invariant under affine group.

Just as in rigid-body mechanics the most natural Hilbert space structures are those based on Haar measure λ on $\mathrm{GL}^+(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, i.e., $d\lambda(\varphi) = (\det\varphi)^{-n} dl(\varphi) = (\det\varphi)^{-n} \varphi^1{}_1 \cdots \varphi^n{}_n$, where l denoting the usual Lebesgue measure on $\mathrm{L}(n, \mathbb{R})$, i.e., the set of all $n \times n$ real matrices and, as a matter of fact, the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. Momentum mappings (Abraham and Marsden, 1978) corresponding to the left and right regular translations (laboratory and co-moving representations) are given by following quantities which may be meaningfully called affine spin (hypermomentum):

$$\Sigma^a{}_b = \frac{\hbar}{i} \varphi^a{}_K \frac{\partial}{\partial \varphi^b{}_K} = \frac{\hbar}{i} \mathbf{L}^a{}_b, \qquad \hat{\Sigma}^A{}_B = \frac{\hbar}{i} \varphi^i{}_B \frac{\partial}{\partial \varphi^i{}_A} = \frac{\hbar}{i} \mathbf{R}^A{}_B. \tag{14}$$

They are formally self-adjoint in $L^2(\mathrm{GL}^+(n, \mathbb{R}), \lambda)$ but not on $L^2(\mathrm{GL}^+(n, \mathbb{R}), l)$. To become such in the latter case they must be completed by some algebraic terms. Obviously, $\Sigma^a{}_b = \varphi^a{}_A \hat{\Sigma}^A{}_B (\varphi^{-1})^B{}_b$, and $\mathbf{L}^a{}_a$, $\mathbf{R}^B{}_B$ are generators of the left and right regular transformations:

$$\Psi((I + \alpha)\varphi) = \Psi(\varphi) + \alpha^i{}_j \mathbf{L}^j{}_i \Psi(\varphi) + o(\alpha), \tag{15}$$

$$\Psi((I + \alpha)\varphi) = \Psi(\varphi) + \alpha^B{}_A \mathbf{R}^A{}_B \Psi(\varphi) + o(\alpha). \tag{16}$$

The skew-symmetric parts are referred to as spin and vorticity (Dyson):

$$\mathbf{S}^i{}_j = \Sigma^i{}_i - \Sigma_j{}^i, \qquad \mathbf{V}^A{}_B = \hat{\Sigma}^A{}_B - \hat{\Sigma}_B{}^A \tag{17}$$

(shift of indices meant in the Kronecker-delta sense). For $n > 2$ the covering group $\overline{\mathrm{GL}^+(n, \mathbb{R})}$ is 2 : 1, and $\mathrm{GL}^+(n, \mathbb{R})$ is doubly-connected.

Remark 3.1. $\overline{\mathrm{GL}^+(n, \mathbb{R})}$ is nonlinear, and so is $\overline{\mathrm{SL}^+(n, \mathbb{R})}$. By "nonlinear" we mean "non-admitting faithful representations in terms of finite-dimensional matrices." The doubly-connected topology of $\mathrm{GL}^+(n, \mathbb{R})$ is seen from the polar decomposition, $GL^+(n, \mathbb{R}) \ni \varphi = UA$, where $U \in SO(n, \mathbb{R})$ and A is symmetric and positively definite. For $n \geq 3$, $SO(n, \mathbb{R})$ is doubly-connected, whereas the manifold of As has evidently the $\mathbb{R}^n$-topology. Topologically, the covering of $GL^+(n, \mathbb{R})$ is given by the Cartesian product $\mathrm{Spin}(n, \mathbb{R}) \times \mathrm{Sym}^+(n, \mathbb{R})$; in the physical case $n = 3$, just $SU(2) \times \mathrm{Sym}^+(3, \mathbb{R})$. And then we identify skew-symmetric tensors with pseudo-vectors:

$$\mathbf{S}^i{}_j = \varepsilon^i{}_j{}^k \mathbf{S}_k, \qquad \mathbf{S}_i = \tfrac{1}{2}\varepsilon_{ij}{}^k \mathbf{S}^j{}_k, \qquad \mathbf{V}^A{}_B = \varepsilon^A{}_B{}^C \mathbf{V}_C,$$
$$\mathbf{V}_A = \tfrac{1}{2}\varepsilon_{AB}{}^C \mathbf{S}^B{}_C. \tag{18}$$

Peter–Weyl expansion on $\overline{GL^+(3, \mathbb{R})}$ gives us: $\Psi(u, A) = \Sigma_s \mathrm{Tr}\,(c^s(A) D^s(u))$, where s are integers and half-integers starting from 0. If Ψ is to be admissible as a probabilistically interpretable wave function on $GL^+(n, \mathbb{R})$, then again the "superselection" rule must be satisfied, namely, (i) only half-integer s are admitted in the series and Ψ is doubly-valued in $GL^+(n, \mathbb{R})$, (ii) only integer s are admitted and Ψ (the more so $\bar{\Psi}\Psi$) is single-valuated. Moreover, no superposition between (i) and (ii) is admitted if Ψ is to be statistically interpretable in $GL^+(n, \mathbb{R})$. So, again the "boson-fermion" superselection rule.

Much more effective, at least in high-symmetry problems, is the two-polar decomposition $GL^+(n, \mathbb{R}) \ni \varphi = LDR^{-1}$, where $L, R \in SO(n, \mathbb{R})$, and D is diagonal; it is convenient to write: $D_{aa} = Q^a = \exp(q^a)$. Then φ is represented by $(L, D, R) \in SO(n, \mathbb{R}) \times \mathbb{R}^n \times SO(n, \mathbb{R})$; however, unlike the polar decomposition, this one is charged with some singularities and non-uniqueness (although not very embarrassing when carefully treated). The Haar and Lebesgue measure λ, l are then represented as follows (Barut and Rączka, 1977):

$$d\lambda(\varphi) = d\lambda\,(L, q, R) = P_\lambda(q)\, d\mu(L)\, d\mu(R)\, dq^1 \cdots dq^n, \tag{19}$$

$$dl(\varphi) = dl\,(L, Q, R) = P_l(Q)\, d\mu(L)\, d\mu(R)\, dQ^1 \cdots dQ^n, \tag{20}$$

where μ is the Haar measure on $SO(n, \mathbb{R})$ and

$$P_\lambda(q) = \prod_{i \neq j} |\sinh(q^i - q^j)|, \quad P_l(Q) = \prod_{i \neq j} |(Q^i - Q^j)(Q^i + Q^j)|. \tag{21}$$

$\mathbf{S}^i{}_j$, $\mathbf{V}^A{}_B$ generate left $SO(n, \mathbb{R})$-regular translations of L, R-factors. Right regular translations are generated respectively by $\rho^a{}_b = (L^{-1})^a{}_i \mathbf{S}^i{}_j L^j{}_b$ and $\tau^a{}_b = (R^{-1})^a{}_A \mathbf{V}^A{}_B R^B{}_b$. As usual, for $n = 3$ we represent them as follows:

$$\rho^a{}_b = \epsilon^a{}_b{}^c \rho_c, \qquad \tau^a{}_b = \epsilon^a{}_b{}^c \tau_c, \qquad \rho_a = \frac{1}{2}\epsilon_{ab}{}^c \rho^b{}_c, \qquad \tau_a = \frac{1}{2}\epsilon_{ab}{}^c \tau^b{}_c. \tag{22}$$

To deal with the doubly-valued functions, i.e., with the covering manifold, we begin with $SU(2) \times \mathbb{R}^3 \times SU(2)$ as an auxiliary tool. The Peter–Weyl theorem gives us the following expansion:

$$\Psi(u, q, v) = \sum_{s,j} \sum_{m,n=-s}^{s} \sum_{k,l=-j}^{j} D^{s}{}_{mn}(u) f^{sj}_{\substack{nk\\ml}}(q) D^{j}{}_{kl}\left(v^{-1}\right), \tag{23}$$

or for eigenstates of $\mathbf{S}_3$, $\mathbf{V}_3$ with $\hbar m$, $\hbar l$-eigenvalues:

$$\Psi^{sj}_{ml}(u, q, v) = \sum_{n=-s}^{s} \sum_{k=-j}^{j} D^{s}{}_{mn}(u) f^{sj}_{nk}(q) D^{j}{}_{kl}(v^{-1}). \tag{24}$$

Obviously, here $\hbar^2 s(s+1)$ and $\hbar^2 j(j+1)$ are eigenvalues of $\mathbf{S}$- and $\mathbf{V}$-Casimirs. But $\mathrm{SU}(2) \times \mathbb{R}^3 \times \mathrm{SU}(2)$ is not diffeomorphic with $\overline{\mathrm{GL}^+(3, \mathbb{R})}$. One can show that the earlier expressions are well-defined wave functions on $\overline{\mathrm{GL}^+(3, \mathbb{R})}$, i.e., "good" doubly-valued wave functions on $\mathrm{GL}(3, \mathbb{R})$ if $(j - s)$ is an integer, i.e., j and s have the same "halfness." Besides, some additional conditions must be satisfied to take into account that the two-polar decomposition of $\mathrm{GL}^+(3, \mathbb{R})$ in terms of $\mathrm{SO}(3, \mathbb{R}) \times \mathbb{R}^3 \times \mathrm{SO}(3, \mathbb{R})$ is non-unique (Sławianowski, 1982, 1988). The earlier wave functions are single-valued in $\mathrm{GL}(3, \mathbb{R})$, when s and j are integers. So, $\Sigma_{s,j:(j-s)\in\mathbb{Z}}$ and $\Sigma_{s,j\in\mathbb{N}\cup\{0\}}$ are well defined respectively on $\overline{\mathrm{GL}(n, \mathbb{R})}$ and $\mathrm{GL}^+(n, \mathbb{R})$. And again there is the "superselection": the latter sum can not be combined with $\Sigma_{s,j}$, where $s = m + 1/2$, $j = n + 1/2$, and m, n are non-negative integers.

We are interested in affinely-invariant geodetic models, i.e., in free affine top. Let us stress, however, that strictly speaking purely geodetic model would be non-physical because it would predict the non-limited dilatational expansion and collapse (although in the infinite time). Therefore, the logarithmic dilatational parameter $q = (q^1 + q^2 + \cdots + q^n)/n$ ($n = 3$ in the physical case; sometimes $n = 2$ or $n = 1$) must be stabilized by some simple model potential $V(q)$, e.g., harmonic oscillator $V_{\mathrm{osc}} = (\kappa/2)q^2$ or the infinite potential well. It turns out that the incompressible (thus applicable in nuclear and hadronic dynamics) geodetic $\mathrm{SL}(n, \mathbb{R})$-models are realistic both on the classical and quantum level and may predict bounded vibrating behaviour (and below-threshold discrete spectrum in quantum theory). In analogy to the spherical rigid body we can postulate the left and right invariant kinetic energy on $\mathrm{GL}^+(n, \mathbb{R})$. On the classical level it would be given by the Casimir expression:

$$T = \frac{A}{2}\mathrm{Tr}(\Omega^2) + \frac{B}{2}(\mathrm{Tr}\Omega)^2 = \frac{A}{2}\mathrm{Tr}(\hat{\Omega}^2) + \frac{B}{2}(\mathrm{Tr}\hat{\Omega})^2, \tag{25}$$

where Ω, $\hat{\Omega}$ are Lie-algebraic objects, just the affine counterparts of the laboratory and co-moving representations of the angular velocity,

$$\Omega = \frac{d\varphi}{dt}\varphi^{-1}, \qquad \hat{\Omega} = \varphi^{-1}\frac{d\varphi}{dt} = \varphi^{-1}\Omega\varphi. \tag{26}$$

The corresponding Laplace–Beltrami operator is expressed in the two-polar terms as follows (in n dimensions):

$$\mathbf{T}^{\mathrm{aff-aff}} = -\frac{\hbar^2}{2A}\mathbf{D}_\lambda + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2}{\partial q^2} + \frac{1}{32A}\sum_{a,b}\frac{\left(\rho^a{}_b - \tau^a{}_b\right)^2}{\sinh^2\frac{q^a-q^b}{2}} - \frac{1}{32A}\sum_{a,b}\frac{\left(\rho^a{}_b + \tau^a{}_b\right)^2}{\cosh^2\frac{q^a-q^b}{2}}.$$

The differential operator $\mathbf{D}_\lambda$ is given by the following expression:

$$\mathbf{D}_\lambda = \frac{1}{P_\lambda}\sum_a \frac{\partial}{\partial q^a} P_\lambda \frac{\partial}{\partial q^a}. \tag{27}$$

This kinetic energy is not positively definite, but its negative term may encode the attraction (strange "centrifugal" attraction) of deformation invariants, whereas its positive counterpart describes the repulsive forces (infinite at coincidence situation). Their balance leads on the classical level to nonlinear elastic vibrations with an open subset of bounded trajectories and an open subset of non-bounded ("dissociated") ones; everything, of course, under the assumption of approximate incompressibility, when some dilatation-stabilizing potential $V(q)$ is used. On the quantum level this means that both the discrete spectrum and the higher-placed continuous one do occur.

For certain reasons it may be convenient to discuss models $\mathbf{T}^{\mathrm{met-aff}}$ invariant under spatial rotations (left translations by orthogonal elements) and right affine transformations, and also conversely, the models $\mathbf{T}^{\mathrm{aff-met}}$ with opposite symmetry properties. Classically:

$$T^{\mathrm{met-aff}} = \frac{I}{2}\mathrm{Tr}(\Omega^T\Omega) + \frac{A}{2}\mathrm{Tr}(\Omega^2) + \frac{B}{2}(\mathrm{Tr}\Omega)^2, \tag{28}$$

$$T^{\mathrm{aff-met}} = \frac{I}{2}\mathrm{Tr}(\hat{\Omega}^T\hat{\Omega}) + \frac{A}{2}\mathrm{Tr}(\hat{\Omega}^2) + \frac{B}{2}(\mathrm{Tr}\hat{\Omega})^2. \tag{29}$$

The first one is a discretization of the Arnold model of ideal fluid as a Hamiltonian system on the group of volume-preserving diffeomorphisms. The second one does not obey the spatial metric relations like, e.g., electrons in crystals, for which the metric tensor is replaced by the effective mass tensor; similar things happen in the theory of defects in solids. After quantization:

$$\mathbf{T}^{\mathrm{met-aff}} = \mathbf{T}^{\mathrm{aff-aff}}[A \mapsto I + A] + \frac{I}{2\left(I^2 - A^2\right)}\|\mathbf{S}\|^2, \tag{30}$$

$$\mathbf{T}^{\mathrm{aff-met}} = \mathbf{T}^{\mathrm{aff-aff}}[A \mapsto I + A] + \frac{I}{2\left(I^2 - A^2\right)}\|\mathbf{V}\|^2. \tag{31}$$

The shorthand $A \mapsto I + A$ means obviously "with A replaced by $I + A$"; $\|\mathbf{S}\|^2$ and $\|\mathbf{V}\|^2$ are squared magnitudes of spin and vorticity, i.e., Casimirs:

$$\|\mathbf{S}\|^2 = -\frac{1}{2}\mathbf{S}^a{}_b\mathbf{S}^b{}_a, \qquad \|\mathbf{V}\|^2 = -\frac{1}{2}\mathbf{V}^A{}_B\mathbf{V}^B{}_A. \tag{32}$$

For the proper choice of I, A, B, these kinetic energies are positively definite. For the dynamically non-affine but physically-justified macroscopically-elastic models with double isotropy, $T^{\mathrm{d'A}} = (I/2)\mathrm{Tr}\left(\dot{\varphi}^T\dot{\varphi}\right)$, we have

$$\mathbf{T}^{\mathrm{d'A}} = -\frac{\hbar^2}{2I}\mathbf{D}_l + \frac{1}{8I}\sum_{a,b}\frac{\left(\rho^a{}_b - \tau^a{}_b\right)^2}{(Q^a - Q^b)^2} + \frac{1}{8I}\sum_{a,b}\frac{\left(\rho^a{}_b + \tau^a{}_b\right)^2}{(Q^a + Q^b)^2}, \tag{33}$$

where

$$\mathbf{D}_l = \frac{1}{P_l}\sum_a \frac{\partial}{\partial q^a}P_l\frac{\partial}{\partial q^a}. \tag{34}$$

Without potential, the earlier geodetic model is non-physical. There are only purely decaying, scattering motions. With some well-adapted potentials invariant under left and right orthogonal translations such a model is useful in macroscopic and molecular problems without, however, any advantage typical for invariant geodetic systems on groups.

For geodetic models and, more generally, for the doubly-isotropic potential models, $V = V(q^1, \ldots, q^n)$ (including those $\mathrm{SL}(n, \mathbb{R})$-geodetic with $V(q)$ stabilizing dilatations), the quantities $\mathbf{S}^2 = \varrho^2$, $\mathbf{V}^2 = \tau^2$ are constants of motion, and s, j are good quantum numbers. Then, for fixed s, j the stationary Schrödinger equation splits into the family of reduced Schrödinger equations for matrix amplitudes f^{sj} depending only on deformation invariants $q^1, \ldots, q^n$ (by deformation invariants one means, in general, the functions of the matrix φ invariant under left and right regular translations by the orthogonal group $\mathrm{SO}(n, \mathbb{R})$); the dependence on angles is algebraized: $\mathbf{H}^{sj} f^{sj} = E^{sj} f^{sj}$. For the affine–affine model the reduced Hamiltonian has the form

$$\begin{aligned}\mathbf{H}^{sj}_{\mathrm{aff-aff}} f^{sj} = & -\frac{\hbar^2}{2A}\mathbf{D}f^{sj} + \frac{\hbar^2 B}{2A(A+nB)}\frac{\partial^2 f^{sj}}{\partial q^2} + V(q^a)f^{sj} \\ & + \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^j_{ab}} - \overrightarrow{S^s_{ab}}\right)^2}{\sinh^2\frac{q^a - q^b}{2}}f^{sj} - \frac{1}{32A}\sum_{a,b}\frac{\left(\overleftarrow{S^j_{ab}} + \overrightarrow{S^s_{ab}}\right)^2}{\cosh^2\frac{q^a - q^b}{2}}f^{sj},\end{aligned}$$

where $\overleftarrow{S^j_{ab}} f^{sj} := f^{sj} S^j_{ab}$, $\overrightarrow{S^s_{ab}} f^{sj} := S^s_{ab} f^{sj}$. The symbols s, j suggest the dimension $n = 3$ and the usual range of these quantum numbers. Nevertheless, the formula may be meant for the general n, then s, j simply run over the set of labels of irreducible unitary representations of $\mathrm{SO}(n, \mathbb{R})$, and S^s_{ab}, S^j_{ab} are basic (hermitian) generators of these representations (9). For $n = 3$ they are simply the standard Wigner matrices of angular momentum. The potential V is necessary only for stabilization or constraining dilatations; on SL$(n, \mathbb{R})$ the potential-free geodetic model is satisfactory.

For the metric-affine model the reduced Hamiltonian is given by

$$\mathbf{H}^{sj}_{\text{met}-\text{aff}} = \mathbf{H}^{sj}_{\text{aff}-\text{aff}}[A \mapsto I + A] + \frac{I}{2(I^2 - A^2)}\hbar^2 C(2, s), \tag{35}$$

where $-C(2, s)$ is the eigenvalue of the second-order Casimir invariant built of generators of regular translations on SO$(n, \mathbb{R})$ in the sth representation:

$$\frac{1}{2}\mathbf{L}^a{}_b \mathbf{L}^b{}_a D^s = C(2, s) D^s, \quad \text{i.e.,} \quad \frac{1}{2}\sum_{a,b} S^s_{ab} S^s_{ba} = \hbar^2 C(2, s) I_N, \tag{36}$$

where I_N denotes the $N \times N$ unit matrix, and N is the dimension of the sth irreducible representation of SO$(n, \mathbb{R})$. Obviously, in the interesting physical case $n = 3$, $N = 2s + 1$, $s = 0, 1/2, 1, \ldots$, and $C(2, s) = s(s + 1)$.

For the affine–metric model we have

$$\mathbf{H}^{sj}_{\text{aff}-\text{met}} = \mathbf{H}^{sj}_{\text{aff}-\text{aff}}[A \mapsto I + A] + \frac{I}{2(I^2 - A^2)}\hbar^2 C(2, j). \tag{37}$$

In a sense, $\mathbf{H}^{sj}_{\text{aff}-\text{aff}}$ and occurrence of additional terms proportional to $\hbar^2 C(2, s)$, $\hbar^2 C(2, j)$ is extremely interesting and seems to be confirmed by the nuclear and hadronic experimental data respectively as the angular momentum and isospin terms. In the incompressible case, when $B = 0$, the quantized geodetic model (without potential) is sufficient for predicting both the discrete and continuous spectrum (bounded and decaying situations). In the compressible case, dilatations must be stabilized by some model potential $V(q)$. Appearing of the discrete and continuous spectra is controlled by the interplay between s and j quantum numbers (they are good quantum numbers corresponding to quantum constants of motion).

The appearance of the formal similarity of the above expressions to integrable lattices formulas is not accidental and may be helpful in the analysis of Sutherland and Calogero-Moser lattices.

4. SOME FINAL REMARKS

Linear group GL$(3, \mathbb{R})$ has been used in nuclear physics as the group which rules geometry of the collective degrees of freedom in the droplet model of nuclei (Bohr and Mottelson, 1975; Rosensteel and Troupe, 1998). However, it was not there the group of dynamical symmetries preserving the Hamiltonian. There are models where GL$(3, \mathbb{R})$ is the so-called non-invariance group. We suggest models which seem to be viable and use GL$(3, \mathbb{R})$ as the group of dynamical symmetries.

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